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THE LOWER LIMITS OF THE LIMIT LOADS OF PERFECTLY PLASTIC STRUCTURES†

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A general method of obtaining the lower limits for limit loads is described, the essentials of which are as follows. The load acting on a structure is represented in the form of a functional series in powers of certain fundamental loads, for each of which, applied separately, the limit load coefficient (LLC) or a lower estimate for it is found. Using those estimates, the corresponding statically admissible stress fields and the coefficients of the expansion in the functional series, the lower limit is found for the LLC for the initial load distribution. Using the system of lower limits of the LLC for fundamental loads, a lower limit of the LLC for any distribution, including an alternating-sign distribution, can be found. This can be very useful when the load distribution on the structure changes with time, as in the case of exposure to snow or wind.

MOST studies of limit loads ([1–5] and elsewhere) are devoted to the formulation and use of the kinematic method of the theory of limiting equilibrium, which gives an upper limit for the limit load. Finding lower limits is a more difficult problem, owing to the fact that the statically admissible stress fields constructed at each point of the structure must exceed the plasticity condition. Verification of this point-by-point inequality is a separate problem for each stress field, requiring either analytical estimates or numerical calculations. The various approaches used for structures of different types ([6–12], etc.) have proved to be very time-consuming and not as simple as the kinematic method. Many of the estimates ([13–16], etc.) merely reduce to choosing statically admissible stress fields in particular cases. Only a lower limit, however, can provide an estimate for the safe load of a structure.

1. Let the stressed state of a rigid plastic or an elastic perfectly plastic structure be described by generalized stresses $\mathbf{Q}$, defined in a certain region S [17]. Each component $\mathbf{F}_\nu$ of the generalized load $\mathbf{F} = (\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_p)$ acting on the structure is the distribution vector of a certain part of the generalized load over the corresponding surface or line D_ν ($\nu = 1, 2, \ldots, p$) (some of which might be due to bulk forces).

Suppose that there is a representation of the initial load in series form

$$\mathbf{F} = \sum_{m=1}^{\infty} \gamma_m \mathbf{F}^{(m)}, \quad \gamma_m \geqslant 0; \quad \mathbf{F}^{(m)} = (\mathbf{F}_1^{(m)}, \ldots, \mathbf{F}_p^{(m)}) \tag{1.1}$$

The vector functions $\mathbf{F}^{(m)}$ are defined on the same sets D_ν ($\nu = 1, 2, \ldots, p$), where the series is component-wise weakly convergent to $\mathbf{F}$ in $L^2(D_\nu)$. For each $M = 1, 2, \ldots$, we consider the partial sum of the series $\mathbf{F}_M = \gamma_1\mathbf{F}^{(1)} + \gamma_2\mathbf{F}^{(2)} + \ldots + \gamma_M\mathbf{F}^{(M)}$ and denote by α, $\alpha^{(m)}$, α_M the LLC for separately applied loads $\mathbf{F}$, $\mathbf{F}^{(m)}$ and $\mathbf{F}_M$, respectively.

Suppose that for each of the generalized loads $\mathbf{F}^{(m)}$ applied separately, a lower limit for the LLC has been found: $\alpha^{(m)} \geqslant \alpha_0^{(m)}$; it might be that the exact value of $\alpha^{(m)}$ is known. Then there exists a generalized stress field $\mathbf{Q}_0^{(m)}$ which balances the load $\alpha_0^{(m)}\mathbf{F}^{(m)}$ and does not exceed the yield point: $f(\mathbf{Q}_0^{(m)}) \leqslant 1$. Here f is a convex stress function, positively uniform of the first degree, characterizing the form of the plasticity condition.

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We write

$$\alpha_M^\circ = \left[\sum_{m=1}^{M} \frac{\gamma_m}{\alpha_0^{(m)}} \right]^{-1}, \quad \alpha_M^* = \frac{\alpha_M^\circ}{\sup_S f(\mathbf{Q}_M^\circ)}; \quad \mathbf{Q}_M^\circ = \sum_{m=1}^{M} \frac{\gamma_m}{\alpha_0^{(m)}} \mathbf{Q}_0^{(m)}$$

It is clear that $\alpha_M^* \geq \alpha_M^\circ$ and the stress field $\alpha_M^* \mathbf{Q}_M^\circ$ is statically admissible for the load $\alpha_M^* \mathbf{F}_M$ (the equilibrium equations are assumed to be linear). It therefore yields a lower limit for the LLC α_M for $\mathbf{F}_M$

$$\alpha_M^\circ \leqslant \alpha_M^* \leqslant \alpha_M \tag{1.2}$$

We will write the principle of virtual power for generalized real stresses $\mathbf{Q}$, strain rates $\mathbf{q}$ and translational velocities $\mathbf{v}$ in a state of collapse corresponding to the load of $\alpha\mathbf{F}$

$$\int_S \mathbf{Q}\mathbf{q}\, dS = \alpha \sum_{\nu=1}^{p} \int_{D_\nu} \mathbf{F}\mathbf{v}\, dD_\nu \tag{1.3}$$

We now write the principle of virtual power with respect to real generalized stresses $\mathbf{Q}_M$ in a state of collapse corresponding to the load $\alpha_M \mathbf{F}_M$ and the velocity field $\mathbf{v}$

$$\int_S \mathbf{Q}_M \mathbf{q}\, dS = \alpha_M \sum_{\nu=1}^{p} \int_{D_\nu} \mathbf{F}_M \mathbf{v}\, dD_\nu \tag{1.4}$$

From the local maximum principle [18] we have $\mathbf{Q}_M\mathbf{q} \leqslant \mathbf{Q}\mathbf{q}$, and thus from (1.3) and (1.4)

$$\alpha \geqslant \alpha_M \left[\left(\sum_{\nu=1}^{p} \int_{D_\nu} \mathbf{F}_M \mathbf{v}\, dD_\nu \right) / \left(\sum_{\nu=1}^{p} \int_{D_\nu} \mathbf{F}\mathbf{v}\, dD_\nu \right) \right] \tag{1.5}$$

Owing to weak component-wise convergence of the sequence $\mathbf{F}_M \to \mathbf{F}$ as $M \to \infty$, the limit of the ratio written in square brackets exists and is equal to one. Taking the upper limit [19] as $M \to \infty$ in (1.15) (the ordinary limit might not exist), we have $\alpha \geqslant \overline{\lim}\, \alpha_M$.

Finally, from (1.2), we obtain the limit

$$\alpha_0 \leqslant \alpha_* \leqslant \alpha \tag{1.6}$$

$$\alpha_* = \overline{\lim}\, \alpha_M^*; \quad \alpha_0 = \overline{\lim}\, \alpha_M^\circ = \lim \alpha_M^\circ = \left(\sum_{m=1}^{\infty} \frac{\gamma_m}{\alpha_0^{(m)}} \right)^{-1} \quad (M \to \infty)$$

Thus, to find a lower limit for the LLC α_0 for load (1.1), only the values of the constants γ_m and estimates $\alpha_0^{(m)}$ of the LLC are needed for each load $\mathbf{F}^{(m)}$ applied separately. To obtain a more accurate limit α_*, we also need information on the corresponding statically admissible stress fields $\mathbf{Q}_0^{(m)}$, and the accuracy of this estimate depends, in turn, on the choice of those fields. The stress fields $\mathbf{Q}_0^{(m)}$ can be constructed directly, by fixing all but a few parameters and then maximizing the lower limit of the LLC with respect to those parameters. Furthermore, the solutions of simpler or known problems, as well as those of similar problems, such as those of the theory of elasticity, can also be used.

As a special case, it follows from (1.6) that if the loads $\mathbf{F}^{(m)}$ $(m = 1, 2, \ldots)$ are limit loads, then the load (1.1) is safe or limiting for

$$\sum_{m=1}^{\infty} \gamma_m \leqslant 1, \quad \gamma_m \geqslant 0$$

2. We will consider the problem of finding the LLC for a prismatic rod for the joint action of an axial force T and a moment M (the two loads increase in proportion to the same parameter). Let Σ_{ij} be the components of the stress tensor in the Cartesian system of coordinates X_i $(i, j = 1, 2, 3)$, with X_3 axis along the rod. We introduce dimensionless variables $x_i = X_i/d$, $\sigma_{ij} = \Sigma_{ij}/\sigma_s$, $t = T/(d^2\sigma_s)$, $m = M/(d^3\sigma_s)$, where d is the characteristic length, and σ_s is the yield point. As usual $\sigma_{11} = \sigma_{22} = \sigma_{12} = 0$, and the other components of the stress tensor are subject to the von Mises yield condition $\sigma_{33}^2 + 3(\sigma_{13}^2 + \sigma_{23}^2) \leqslant 1$.

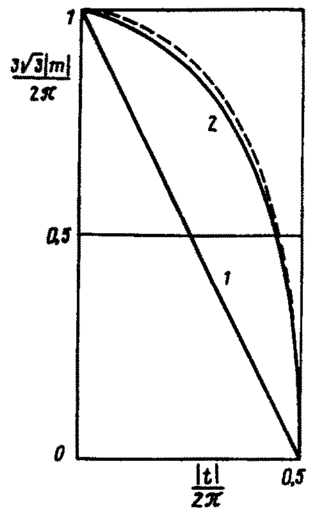

FIG. 1.

For a rod of circular cross-section S, with circumference $x_1^2+x_2^2=1$ (in the case here d is dimensionless radius of the rod) when only one of the loads is acting the LLC are ([20], p. 69): $\alpha_1^\circ=\pi|t|^{-1}$, $\alpha_2^\circ=2\pi(3\sqrt{3}|m|)^{-1}$. When an axial force t and a torque m are acting jointly taking $\gamma_1=\gamma_2=1$, $\gamma_3=\gamma_4=\ldots=0$ in (1.1) and (1.2) we obtain the estimate $\alpha\geqslant\alpha_0=2\pi(2|t|+3\sqrt{3}|m|)^{-1}$. The corresponding statically admissible stress fields

$$\sigma_{33}^{(1)}=\operatorname{sign} t,\quad \sigma_{ij}^{(1)}=0\quad (i+j<6),\quad \sigma_{12}^{(2)}=\sigma_{ii}^{(2)}=0$$
$$\sigma_{13}^{(2)}=-x_2R,\quad \sigma_{23}^{(2)}=x_1R,\quad R=\operatorname{sign} m\,[3(x_1^2+x_2^2)]^{-1/2}$$

yield the estimate

$$\alpha\geqslant\alpha_*=\alpha_0\inf_S[\sigma_{33}^2+3(\sigma_{13}^2+\sigma_{23}^2)]^{-1/2}$$
$$\sigma_{ij}=(\alpha_0/\alpha_1^\circ)\sigma_{ij}^{(1)}+(\alpha_0/\alpha_2^\circ)\sigma_{ij}^{(2)}$$

In this case, clearly,

$$\alpha_*=\alpha_1^\circ\alpha_2^\circ(\alpha_1^\circ+\alpha_2^\circ)^{-1/2}=2\pi(4t^2+27m^2)^{-1/2}$$

Curves 1 and 2 in Fig. 1 are the boundaries of the range of statically admissible values of t and m corresponding, respectively, to LLC α_0 and α_*, i.e. the factors $\{(\beta t,\beta m)\colon 0\leqslant\beta\leqslant\alpha_0\}$ and $\{(\beta t,\beta m)\colon 0\leqslant\beta\leqslant\alpha_*\}$. The result for α_* is practically the same as the exact solution (the limit curve [20, p. 53] is the dashed line).

For a rod of rectangular cross-section S bounded by straight lines $2|x_1|=1$, $2|x_2|=\eta$ $(\eta\leqslant1)$, when only one of the loads is acting, the LLC are [20, p. 70]

$$\alpha_1^\circ=\eta|t|^{-1},\quad \alpha_2^\circ=\eta^2(3-\eta)(12\sqrt{3}|m|)^{-1}$$

The statically admissible stress fields have the form ($\kappa=\pm1$)

$$\sigma_{33}^{(1)}=\operatorname{sign} t,\quad \sigma_{ij}^{(1)}=0\quad (i+j<6),\quad \sigma_{12}^{(2)}=\sigma_{ii}^{(2)}=0$$

$$\sigma_{13}^{(2)}=\begin{cases}0\\ -\kappa/\sqrt{3}\\ 0\\ \kappa/\sqrt{3}\end{cases}\qquad \sigma_{23}^{(2)}=\begin{cases}\kappa\sqrt{3}\text{ in triangle } M_1OM_2\\ 0\quad\text{in trapezium } M_2O_1O_2M_3\\ -\kappa/\sqrt{3}\quad\text{in triangle } M_3OM_4\\ 0\quad\text{in trapezium } M_4O_2O_1M_1\end{cases}$$

(Fig. 2). Then, as before, we have the limits

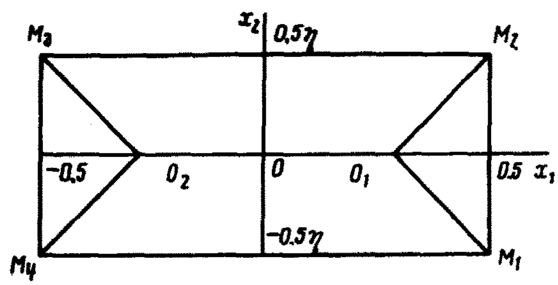

FIG. 2.

$$\alpha \geqslant \alpha_0 = \eta^2(3-\eta)[\eta(3-\eta)\,|t| + 12\sqrt{3}\,|m|]^{-1}$$
$$\alpha \geqslant \alpha_* = \alpha_1^\circ \alpha_2^\circ (\alpha_1^\circ + \alpha_2^\circ)^{-1/2} = \eta^2(3-\eta)[\eta^2(3-\eta)^2 t^2 + 432 m^2]^{-1/2}$$

If the coefficients of $|m|$ and $|t|$ in Fig. 1 are replaced, respectively, by $12\sqrt{3}/[\eta^2(3-\eta)]$ and $1/(2\eta)$, curves 1 and 2 will correspond to LLC α_0 and α_* in this case; there is no exact solution of the problem.

3. Consider a circular or annular plate $0 \leqslant a \leqslant r \leqslant b$ of constant thickness $2h$ subjected to a distributed transverse bending load $p(r)$. Here $p(r)$ is a certain Riemann-integrable function, which is of alternating sign in the general case, and r is the radial coordinate. We introduce dimensionless variables $q = pb^2 M_0^{-1}$, $x = r/b$, $\eta = a/b$, where $M_0 = \sigma_s h^2$.

We define the sequence of partitions P_n: $\eta = x_0^{(n)} \leqslant x_1^{(n)} \leqslant \ldots \leqslant x_{k_n}^{(n)} = 1$ $(n = 1, 2, \ldots)$ such that

$$\lim_{n\to\infty} \max_{1 \leqslant i \leqslant n} \Delta x_i^{(n)} = 0, \qquad \Delta x_i^{(n)} = x_i^{(n)} - x_{i-1}^{(n)}$$

and in the segment $[\eta, 1]$ we consider the step function ψ_n which, on $\langle x_{i-1}^{(n)}, x_i^{(n)}\rangle$, takes the value $q(t_i^{(n)})$, where $t_i^{(n)} \in [x_{i-1}^{(n)}, x_i^{(n)}]$. The values of the function at the ends of the interval are arbitrary. It can be shown that the sequence ψ_n converges to $q(x)$ with respect to the norm L^2 (and, furthermore, converges with respect to the norm L^1).

Suppose that a lower limit for the LLC ($\alpha^\circ(z_1, z_2)$ is known in the case where the distribution of the dimensionless load is identical with the characteristic function $K(z_1, z_2, x)$ of any segment $[z_1, z_2] \subset [\eta, 1]$: $K(z_1, z_2, x) = 1$ if $x \in [z_1, z_2]$, and $K(z_1, z_2, x) = 0$ otherwise. Under the Tresca and the von Mises yield conditions $\alpha^\circ(z_1, z_2)$ is also the lower limit for the load distribution $-K(z_1, z_2, x)$. Then for the step function

$$\psi_n = \sum_{i=1}^{k_n} q(t_i^{(n)}) K(x_{i-1}^{(n)}, x_i^{(n)}, x)$$

from (1.6) we obtain the following limit for the LLC

$$\alpha_n \geqslant \alpha_n^\circ = \left[\sum_{i=1}^{k_n} \frac{|q(t_i^{(n)})|}{\alpha^\circ(x_{i-1}^{(n)}, x_i^{(n)})}\right]^{-1} \tag{3.1}$$

The corresponding limit for an arbitrary initial load distribution has the form

$$\alpha \geqslant \alpha_0 = \overline{\lim_{n\to\infty}}\, \alpha_n^\circ.$$

Consider, for example, a uniform circular spherically supported plate $0 = \eta \leqslant x \leqslant 1$ under the Tresca yield condition. In this case [20, p. 114] the exact value of the LLC is

$$\alpha^\circ(z_1, z_2) = 6[3(z_2^2 - z_1^2) - 2(z_2^3 - z_1^3)]^{-1}$$

Substituting this value into (3.1) and using the definition of a Riemann integral, we obtain the limit for the LLC for an arbitrary load distribution

$$\alpha \geqslant \alpha_0 = \left(\int_0^1 |q(x)| x(1-x)\, dx \right)^{-1} \tag{3.2}$$

In the case where the function $q(x)$ has constant sign, α_0 is the exact value of the LLC [21], and the equality applies in (3.2).

Suppose, for example, that $q(x) = q_1 + (q_2 - q_1)x$ (q_1 and q_2 are constants); to fix our ideas take $q_1 > 0$, $q_2 < 0$. The limit (3.2) takes the form

$$\alpha \geqslant 12(q_1 - q_2)^3 (q_1^4 - 2q_1^3 q_2 - 2q_1 q_2^3 + q_2^4)^{-1}$$

For a stepped load

$$q(x) = \sum_{i=1}^{n} q_i K(a_i, b_i, x), \qquad a_1 \leqslant b_1 \leqslant a_2 \leqslant \ldots \leqslant b_n$$

and from (3.2)

$$\alpha \geqslant 6 \left\{ \sum_{i=1}^{n} |q_i| [3(b_i^2 - a_i^2) - 2(b_i^3 - a_i^3)] \right\}^{-1}$$

In the limit, this provides the limit for a set of concentrated annular loads P_i uniformly distributed around the circles

$$x_i = a_i \quad (i = 1, 2, \ldots, n), \qquad \alpha \geqslant 2\pi \left[\sum_{i=1}^{n} |P_i| (1 - a_i) \right]^{-1}$$

4. Consider a hollow shell of constant thickness $2h$, of rectangular cross-section $0 \leqslant X_1 \leqslant a$, $0 \leqslant X_2 \leqslant b$, the entire edge of which is clamped, and which is bent by a transverse load $p(X_1, X_2)$. We assume, as usual, that the middle plane of the shell can be described by the equation

$$2Z = H + K_1 X_1 + K_2 X_2 + K_{11} X_1^2 + 2K_{12} X_1 X_2 + K_{22} X_2^2$$
$$H,\ K_i,\ K_{ij} = \text{const} \quad (i, j = 1, 2)$$

The equations of equilibrium of a shell in dimensionless variables have the form [22]

$$n_{11,1} + n_{12,2} = 0, \qquad n_{12,1} + n_{22,2} = 0 \tag{4.1}$$
$$k_{11} n_{11} + 2k_{12} n_{12} + k_{22} n_{22} + m_{11,11} + 2m_{12,12} + m_{22,22} = -q(x_1, x_2)$$
$$x_i = X_i/a, \quad 0 \leqslant x_1 \leqslant 1, \quad 0 \leqslant x_2 \leqslant \eta = b/a, \qquad k_{ij} = 2a^2 K_{ij}/h$$
$$m_{ij} = M_{ij}/M_0, \quad n_{ij} = N_{ij}/N_0, \quad M_0 = \sigma_s h^2, \quad N_0 = 2\sigma_s h, \quad q = pa^2/M_0$$

(the comma in the subscripts denotes differentiation with respect to the respective dimensionless coordinate). Here k_{ij} are the dimensionless curvatures, M_{ij} and N_{ij} are the moments and shear forces, and σ_s is the yield point for uniaxial tension.

We consider the Hodge plasticity condition [23]

$$H(m) \leqslant 1, \qquad H(n) \leqslant 1 \tag{4.2}$$
$$H(l) = l_{11}^2 - l_{11} l_{22} + l_{22}^2 + 3l_{12}^2; \qquad l = m, n$$

When the edge is clamped, every force and moment field which satisfies Eqs (4.1) and the plasticity condition will be statically admissible.

The system of equations (4.1) reduces to the single equation

$$(m_{11} + k_{22}\varphi)_{,11} + 2(m_{12} - k_{12}\varphi)_{,12} + (m_{22} + k_{11}\varphi)_{,22} = -q(x_1, x_2) \tag{4.3}$$

where the forces in the shell are related to the stress function φ by the equations

$$n_{11} = \varphi_{,22}, \quad n_{22} = \varphi_{,11}, \quad n_{12} = -\varphi_{,12} \tag{4.4}$$

We express the function $q(x_1, x_2)$ in the form of a double Fourier cosine series

$$q(x_1,x_2) = \sum_{n,m=0}^{\infty} \gamma_{nm} \cos n\pi x_1 \cos m\pi \frac{x_2}{\eta} = \sum_{n,m=0}^{\infty} |\gamma_{nm}| q_{nm}$$

We use the notation

$$q_{nm} = s_{nm} \cos n\pi x_1 \cos m\pi \frac{x_2}{\eta}, \qquad p_{nm} = s_{nm} \sin n\pi x_1 \sin m\pi \frac{x_2}{\eta}$$

where $s_{nm} = 1$ if $\gamma_{nm} \geqslant 0$ and $s_{nm} = -1$ otherwise.

We find the limits α°_{nm} for each of the basic loads q_{nm}; we then obtain the corresponding limit of the LLC for the initial load distribution $q(x_1, x_2)$, allowing for convergence of the Fourier series for norm L^2, using the formula

$$\alpha \geqslant \alpha_0 = \left(\sum_{n,m=0}^{\infty} \frac{|\gamma_{nm}|}{\alpha^{\circ}_{nm}} \right)^{-1} \tag{4.5}$$

Thus, we consider the shell subjected to the load $q = A_{nm} q_{nm}$. If $nm \neq 0$, we can represent the distribution of moments and the stress function in the form

$$\begin{aligned} m_{11} &= c_1 q_{nm} + c_2 p_{nm}, \quad & m_{12} &= c_3 q_{nm} + c_4 p_{nm} \\ m_{22} &= c_5 q_{nm} + c_6 p_{nm}, \quad & \varphi &= c_7 q_{nm} + c_8 p_{nm} \end{aligned} \tag{4.6}$$

($c_1, c_2, \ldots, c_8$ are constants). Substituting (4.6) into (4.3) and equating coefficients of p_{nm} and q_{nm} on the left- and right-hand sides, we obtain

$$A_{nm} = \pi^2[n^2(c_1 + k_{22}c_7) - 2nm\eta^{-1}(c_4 - k_{12}c_8) + m^2\eta^{-2}(c_5 + k_{11}c_7)] \tag{4.7}$$

$$0 = n^2\eta^2(c_2 + k_{22}c_8) - 2nm\eta(c_3 - k_{12}c_7) + m^2(c_6 + k_{11}c_8) \tag{4.8}$$

We now substitute (12) and (14) into (10). After similar algebra, we arrive at the system of inequalities

$$A_i q^2_{nm} + B_i p_{nm} q_{nm} + C_i p^2_{nm} \leqslant 1 \quad (i = 1,2) \tag{4.9}$$

Here

$$\begin{aligned} A_1 &= c_1^2 - c_1 c_5 + c_5^2 + 3c_3^2 \geqslant 0 \\ B_1 &= 2c_1c_2 - c_1c_6 + 6c_3c_4 - c_2c_5 + 2c_5c_6, \quad C_1 = c_2^2 - c_2c_6 + c_6^2 + 3c_4^2 \geqslant 0 \\ A_2 &= \beta_1 c_7^2 + \beta_2 c_8^2 \geqslant 0, \quad B_2 = 2(\beta_1 + \beta_2)c_7c_8, \quad C_2 = \beta_1 c_8^2 + \beta_2 c_7^2 \geqslant 0 \\ \beta_1 &= \pi^4(m^4\eta^{-4} - n^2m^2\eta^{-2} + n^4) \geqslant 0, \quad \beta_2 = 3n^2m^2\pi^4\eta^{-2} > 0 \end{aligned}$$

It can be shown that system (4.9) will be satisfied at each point of the middle plane of the shell only if

$$A_1 \leqslant 1, \quad C_1 \leqslant 1, \quad A_2 \leqslant 1, \quad C_2 \leqslant 1 \tag{4.10}$$

From the theory of limiting equilibrium, the best lower estimate of α°_{nm} is obtained by solving the following non-linear programming problem for each pair n, m ($nm \neq 0$): find the maximum value of A_{nm} as a function of $c_1, c_2, \ldots, c_8$, with conditions (4.8) and (4.10).

For a plate ($k_{ij} = 0$), an explicit limit can be found

$$\begin{aligned} \alpha^{\circ}_{nm} &= \pi^2[\sqrt{r^2 + s^2} + 2 \cdot 3^{-1/2} nm\eta^{-1}] \\ r &= n^2 + m^2\eta^{-2}, \quad s = 3^{-1/2}(n^2 - m^2\eta^{-2}) \end{aligned}$$

If $n \neq 0$, $m = 0$, the force and moment field is taken in the form

$$m_{11} = -c_1 q_{n0}, \quad m_{22} = -g(x_2) q_{n0}, \quad m_{12} = -(c_4 x_2 + c_5) p_n, \quad \varphi = -c_6 q_{n0}$$

Then from Eqs (4.4) it follows that $n_{11}=n_{12}=0$, $n_{22}=c_6n^2\pi^2q_{n0}$. After substitution into (4.3), we obtain $A_{n0}=-(k_{22}c_6+c_1)n^2\pi^2+2c_4n\pi+2c_2$. The system of inequalities

$$\Phi(g)\cos^2 n\pi x_1+3(c_4x_2+c_5)^2\sin^2 n\pi x_1\leqslant 1,\quad c_6^2n^4\pi^4\cos^2 n\pi x_1\leqslant 1$$
$$\Phi(g)=c_1^2-c_1g+g^2$$

will be satisfied for all $x_1\in[0,1]$ only if

$$\Phi(g)\leqslant 1,\quad 3(c_4x_2+c_5)^2\leqslant 1,\quad c_6^2n^4\pi^4\leqslant 1 \tag{4.11}$$

The numbers c_3 and $c=c_3+\tfrac{1}{2}c_2\eta^2$ are extremal values of the function $g(x_2)$ in the segment $[0,\eta]$. Since the largest value of the positive quadratic function $\Phi(g)$ is reached at one end of the interval, satisfaction everywhere on the plate of the first of the inequalities (4.11) is equivalent to the system

$$\Phi(c_3)\leqslant 1,\quad \Phi(c)\leqslant 1 \tag{4.12}$$

The second and third inequalities in (4.11) will be satisfied for all $x_2\in[0,\eta]$ only if

$$|c_5|\leqslant 3^{-1/2},\quad |c_4\eta+c_5|\leqslant 3^{-1/2},\quad |c_6|\leqslant(n\pi)^{-2} \tag{4.13}$$

It is clear that if conditions (4.12) and (4.13) hold, then the maximum value of A_{n0} is made up of the maximum of the function $B_{n0}=-c_1n^2\pi^2+8(c-c_3)\eta^{-2}$ under conditions (4.12) and the maximum of the function $2c_4n\pi-k_{22}c_6n^2\pi^2$ under (4.13).

Both maxima are defined explicitly; as a result, we obtain the limit

$$\alpha_{n0}^{\circ}=2(n^4\pi^4\eta^2+192\eta^{-2})(576+3n^4\pi^4\eta^4)^{-1/2}+4\cdot 3^{-1/2}n\pi/\eta$$

A similar limit can be obtained for load q_{0m} $(m\neq 0)$:

$$\alpha_{0m}^{\circ}=2(m^4\pi^4\eta^{-2}+192\eta^2)(576\eta^4+3m^4\pi^4)^{-1/2}+4\cdot 3^{-1/2}m\pi/\eta$$

To find the limit α_{00}°, we take the base load to be a uniformly distributed load of intensity $q_{00}\equiv s_{00}$. In that case, the generalized force field can be written in the form

$$\begin{aligned}
m_{11}&=s_{00}(-c_1x^2+c_2y^2\eta^{-2}+1), & n_{11}&=-s_{00}c_4\\
m_{22}&=s_{00}(-c_1y^2\eta^{-2}+c_2x^2+1), & n_{22}&=-s_{00}c_5\\
m_{12}&=s_{00}c_3xy\eta^{-1}, & n_{12}&=-s_{00}c_6\\
x&=x_1-\tfrac{1}{2},\quad y=x_2-\tfrac{1}{2}\eta
\end{aligned}$$

where $c_1, c_2, \ldots, c_6$ are constants. We find the corresponding load on the shell from the second equation of (4.1)

$$q=A_{00}q_{00},\quad A_{00}=B_{00}+C_{00}$$
$$B_{00}=k_{11}c_4+2k_{12}c_6+k_{22}c_5,\quad C_{00}=2c_1(1+\eta^{-2})-2c_3\eta^{-1}$$

The second inequality of (4.2) in this case has the form

$$c_4^2-c_4c_5+c_5^2+3c_6^2\leqslant 1 \tag{4.14}$$

Then

$$\begin{aligned}
&m_{11}^2-m_{11}m_{22}+m_{22}^2+3m_{12}^2=\\
&=M(x^4+y^4\eta^{-4})-(D_1-3c_3^2)\eta^{-2}x^2y^2+(c_2-c_1)(x^2+y^2\eta^{-2})+1\leqslant\\
&\leqslant(\tfrac{1}{4}M+c_2-c_1)(x^2+y^2\eta^{-2})-(D_1-3c_3^2)\eta^{-2}x^2y^2+1\\
&M=c_1^2+c_1c_2+c_2^2,\quad D_1=c_1^2+4c_1c_2+c_2^2
\end{aligned}$$

Since $|x|\leqslant\tfrac{1}{2}$, $|y|\leqslant\tfrac{1}{2}\eta$, we have $x^4\leqslant\tfrac{1}{4}x^2$, $y^4\leqslant\tfrac{1}{4}y^2\eta^2$.

We equate the following expressions to zero

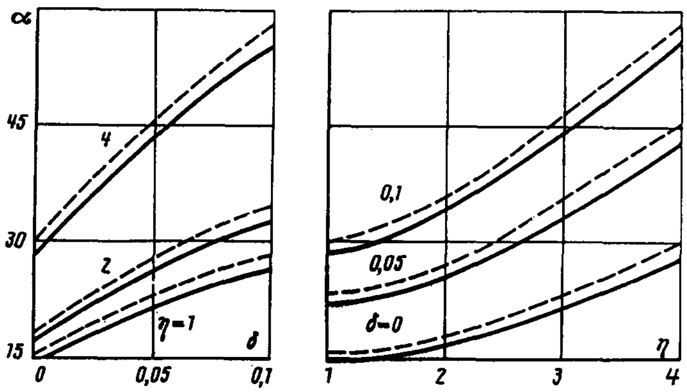

FIG. 3.

$$M + 4(c_2 - c_1) = 0, \quad D_1 - 3c_3^2 = 0 \qquad (4.15)$$

Then the first inequality of (4.2) obviously holds everywhere in the shell.

Thus, the maximum value of A_{00} is made up of the maximum of B_{00} with the condition (4.14) and the maximum of C_{00} under (4.15). It can be shown that the former is equal to

$$2 \cdot 3^{-1/2}(k_1^2 + k_{11}k_{22} + k_{22}^2 + k_{12}^2)^{1/2}$$

We look for the maximum of C_{00} in the case (4.15) as the maximum of a function of one variable

$$C_{00}(c_1) = 2c_1(I + \eta^{-2}) + 2 \cdot 3^{-1/2}\eta^{-1}\sqrt{D_1}$$
$$2c_2 = -4 - c_1 \pm \sqrt{D_2}, \quad D_2 = -3c_1^2 + 24c_1 + 16$$

with $D_1 \geqslant 0$, $D_2 \geqslant 0$, the second inequality being equivalent to $4 - 8 \times 3^{-1/2} \leqslant c_1 \leqslant 4 + 8 \times 3^{-1/2}$. This is an easy problem to solve by numerical methods.

A computer algorithm for the numerical calculation of the lower limits of hollow shells based on the above technique has been written for the EC-1060 computer. A fixed number of terms of the series is used, and the non-linear programming problem of finding $\max A_{nm}$ with constraints (4.8), (4.10) is solved repeatedly, using the NPGLM program based on the method of penalties. Optimal plans obtained can, owing to the known error of the method, be either inside or outside the range defined by (4.8) and (4.10). In any case they should be projected on to the boundary of that range, that is, multiplied by numbers which will project them on to the boundary. A correction of this kind is possible, owing to the linearity of the entire problem and the uniformity of the restraints.

We will estimate the load-carrying capacity of a clamped shell, the middle plane of which is described by the equation

$$Z = H[1 - 2(X_1/a - 1/2)^2 - 2(X_2/b - 1/2)^2]$$

for two different load distributions:

$$q = 1/4\pi^2\eta^{-1}\sin \pi x_1 \sin \pi x_2/\eta \quad \text{(the solid line)}$$

and

$$q = 36\eta^{-2}x_1x_2(1 - x_1)(1 - x_2/\eta) \quad \text{(the dashed line)},$$

corresponding to the same total load on the shell. We take $h/a = 0.02$. The graphs obtained from a numerical calculation of the dependence of the LLC $\alpha = \alpha(\delta)$, $\delta = H/a$ for constant η, as well as $\alpha = \alpha(\eta)$ at constant δ, are shown in Fig. 3. The limit (4.5) converges rapidly as the number of terms retained in (1.1) increases, and an error of less than 1% is obtained by keeping the first nine or ten terms of the series for each coordinate.

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